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On the orthogonal basis of the symmetry classes of tensors associated with certain characters[☆]

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Abstract

A necessary condition is given for the existence of an O-basis for the symmetry classes of tensors associated with a finite group and the irreducible constituents of the permutation character of the finite group. This extends a result of R.R. Holmes [see the Main theorem of Linear and Multilinear Algebra 39 (1995) 241–243]. © 2001 Elsevier Science Inc. All rights reserved.

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1. Introduction

Let V be an m-dimensional vector space over the complex field $\mathbb{C}$. Let $\overset{n}{\otimes} V$ be the nth tensor power of V and write $v_1 \otimes \cdots \otimes v_n$ for the decomposable tensor product of the indicated vectors. To each permutation σ in $\$_n$ there corresponds a unique linear operator $P(\sigma) : \overset{n}{\otimes} V \to \overset{n}{\otimes} V$ determined by $P(\sigma)(v_1 \otimes \cdots \otimes v_n) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}$. Let G be a subgroup of $\$_n$ and let $\mathrm{Irr}(G)$ be the set of all the irreducible

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complex characters of G. It follows from the orthogonality relations for characters that

$$\left\{ T(G,\chi) : \overset{n}{\otimes} V \to \overset{n}{\otimes} V \;\middle|\; T(G,\chi) = \frac{\chi(1)}{|G|} \sum_{\sigma \in G} \chi(\sigma) P(\sigma),\ \chi \in \mathrm{Irr}(G) \right\}$$

is a set of annihilating idempotents which sum to the identity. The image of $\overset{n}{\otimes} V$ under the $T(G,\chi)$ is called the *symmetry class of tensors* associated with G and χ and is denoted by $V_\chi^n(G)$. The image of $v_1 \otimes \cdots \otimes v_n$ under $T(G,\chi)$ is denoted by $v_1 * \cdots * v_n$ and is called a *decomposable symmetrized tensor*.

Let Γ_m^n be the set of all sequences $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $1 \leqslant \alpha_i \leqslant m$. Then the group G acts on Γ_m^n by $\sigma \cdot \alpha = (\alpha_{\sigma^{-1}(1)}, \ldots, \alpha_{\sigma^{-1}(n)})$, where $\sigma \in G$ and $\alpha \in \Gamma_m^n$. Let $O(\alpha) = \{\sigma \cdot \alpha \mid \sigma \in G\}$ be the *orbit* of α, and G_α be its *stabilizer subgroup*, i.e., $G_\alpha = \{\sigma \in G \mid \sigma \cdot \alpha = \alpha\}$, and consider a system Δ of distinct representatives of the orbits of Γ_m^n.

Let $\{e_1, \ldots, e_m\}$ be a basis of V. Denote by e_α^* the tensor $e_{\alpha_1} * \cdots * e_{\alpha_n}$, where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \Gamma_m^n$. For $\alpha \in \Delta$, $V_\alpha^* = \langle e_{\sigma\cdot\alpha}^* \mid \sigma \in G \rangle$ is called the *orbital subspace* of $V_\chi^n(G)$, and we can easily prove that

$$V_\chi^n(G) = \bigoplus_{\alpha \in \Delta} V_\alpha^*. \tag{1}$$

Note that it is possible for some $\alpha \in \Delta$ to have $V_\alpha^* = 0$. But Freese (see [2]) proved that for $\alpha \in \Delta$

$$\dim V_\alpha^* = \frac{\chi(1)}{|G_\alpha|} \sum_{\sigma \in G_\alpha} \chi(\sigma), \tag{2}$$

therefore, if we consider

$$\overline{\Delta} = \left\{ \alpha \in \Delta \;\middle|\; \sum_{\sigma \in G_\alpha} \chi(\sigma) \neq 0 \right\},$$

then the sum in (1) will be reduced to the form

$$V_\chi^n(G) = \bigoplus_{\alpha \in \overline{\Delta}} V_\alpha^*. \tag{3}$$

Of course we define the right-hand side of (3) to be 0, if $\overline{\Delta} = \phi$.

A particular case appears when we assume that V is an m-unitary space. In this case, the inner product on V induces an inner product on $\overset{n}{\otimes} V$, whose restriction to $V_\chi^n(G)$ satisfies

$$\langle u_1 * \cdots * u_n \mid v_1 * \cdots * v_n \rangle = \frac{\chi(1)}{|G|} \sum_{\sigma \in G} \chi(\sigma) \prod_{i=1}^{n} A_{i\sigma(i)},$$

where $A = [A_{ij}]_{n \times n} = [\langle u_i \mid v_j \rangle]_{n \times n}$.

If $\{e_1, \dots, e_m\}$ is an orthonormal basis of V, then the sum appeared in (3) is an orthogonal direct sum. In addition, we can obtain

$$\langle e^*_{g\cdot\alpha} \mid e^*_{g'\cdot\alpha}\rangle = \frac{\chi(1)}{|G|} \sum_{\sigma\in G_\alpha} \chi(g'\sigma g^{-1}). \tag{4}$$

Let W be a subspace of $V^n_\chi(G)$. An orthogonal basis of W of the form $\{e^*_\alpha \mid \alpha \in S\}$, where S is a subset of Γ^n_m, is called an *O-basis* of W. Because the sum appeared in (3) is an orthogonal direct sum, $V^n_\chi(G)$ has an O-basis if and only if for all $\alpha \in \overline{\Delta}$, the orbital subspace V^*_α has an O-basis. Note that, if χ is of degree one, since $\dim V^*_\alpha = 1$ for all $\alpha \in \overline{\Delta}$, then V^*_α has an O-basis for all $\alpha \in \overline{\Delta}$, which implies that $V^n_\chi(G)$ has such a basis.

Several papers are devoted to the investigation of the existence of an O-basis for $V^n_\chi(G)$, see for example [6]. In [4] a necessary and sufficient condition for the existing of an O-basis for $V^n_\chi(G)$ is given, where G is a cyclic or a dihedral group and in [1] when G is a dicyclic group.

The main purpose of this article is to generalize a result of Holmes (see [3, Main theorem]). In fact, he proved that if G is a 2-transitive subgroup of S_n, $n \geqslant 3$, $\chi = \theta - 1_G$, where θ is permutation character of G, and $m \geqslant 2$, then $V^n_\chi(G)$ does not have an O-basis.

In the following section, we will omit condition of "2-transitivity of G" and will find a necessary condition for the existence of an O-basis for the symmetry class of tensors associated with G and any constituent of its permutation character. This generalization will be independent of the permutation structure of the group G.

2. Results

Let V be an m-unitary space, G a finite group, and Ω a set of n elements. Suppose G acts faithfully on Ω, so we can assume that G is a subgroup of S_n. In fact, we consider $\{f_\sigma \mid \sigma \in G\}$ as the group G, where $f_\sigma : \Omega \longrightarrow \Omega$ defined by $f_\sigma(\omega) = \sigma \cdot \omega$ for all $\omega \in \Omega$, is a permutation on n letters. Therefore the inner product space $V^n_\chi(G)$ is meaningful for all $\chi \in \operatorname{Irr}(G)$. Denote the permutation character of G by θ. For $\sigma \in G$, the value $\theta(\sigma)$ is the number of letters fixed by σ, i.e., $\theta(\sigma) = |\{\omega \in \Omega \mid \sigma\cdot \omega = \omega\}|$.

Main Theorem. *Let G be a finite group and let Ω be a set of n elements, $n \geqslant 2$. Assume that G acts transitively and faithfully on Ω and let V be an m-unitary space, $m \geqslant 2$. Let χ be an irreducible constituent of θ, where θ is the permutation character of G. If $\chi(1)(\chi, \theta)_G > \frac{1}{2}n$, then $V^n_\chi(G)$ does not have an O-basis.*

Proof. Let $\{e_1, \dots, e_m\}$ be an orthonormal basis of V. Suppose $V^n_\chi(G)$ has an O-basis. Then, by (3), for all $\alpha \in \overline{\Delta}$, the orbital subspace V^*_α has an O-basis. Put $\alpha =$

$(1, 2, \dots, 2)$, then $n \geqslant 2$ and $m \geqslant 2$ implies that $\alpha \in \Gamma_m^n$. Consider the action of G on Γ_m^n and choose Δ such that $\alpha \in \Delta$.

Without loss of generality, we can assume that $\Omega = \{1, 2, \dots, n\}$ and therefore $G_\alpha = G_1$, where G_α is the stabilizer subgroup of α when G acts on Γ_m^n and G_1 is the stabilizer subgroup of 1 when G acts on Ω. So by Lemma 5.14 of [5] we obtain $1_{G_\alpha} \uparrow^G = 1_{G_1} \uparrow^G = \theta$. Frobenius reciprocity implies that

$$\begin{aligned}\sum_{\sigma \in G_\alpha} \chi(\sigma) &= |G_\alpha|(\chi \downarrow_{G_\alpha}, 1_{G_\alpha})_{G_\alpha} \\ &= |G_\alpha|(\chi, 1_{G_\alpha} \uparrow^G)_G \\ &= |G_\alpha|(\chi, \theta)_G, \end{aligned} \tag{5}$$

so by hypotheses, $(\chi, \theta)_G \neq 0$, and we obtain $\sum_{\sigma \in G_\alpha} \chi(\sigma) \neq 0$ and $\alpha \in \overline{\Delta}$. By the above discussion, V_α^*, $\alpha = (1, 2, \dots, 2)$, has an O-basis. We now have $[G : G_\alpha] = [G : G_1] = |\Omega| = n$, so $G = \bigcup_{i=1}^n g_i G_\alpha$, where $\{g_1, \dots, g_n\}$ is a system of distinct representatives of left cosets of G_α in G.

Since, by (2) and (5),

$$\dim V_\alpha^* = \frac{\chi(1)}{|G_\alpha|} \sum_{\sigma \in G_\alpha} \chi(\sigma) = \chi(1)(\chi, \theta)_G =: s,$$

we can assume that $\{e^*_{g_1\cdot\alpha}, e^*_{g_2\cdot\alpha}, \dots, e^*_{g_s\cdot\alpha}\}$ is an O-basis for V_α^*.

Define the $n \times n$ complex matrix $A = [A_{ij}]$ by

$$A_{ij} = \langle e^*_{g_i\cdot\alpha} \mid e^*_{g_j\cdot\alpha} \rangle.$$

Note that $s < n$, therefore A may be written in the form

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix},$$

where A_1, A_2, A_3, and A_4 are matrices of sizes $s \times s$, $s \times (n-s)$, $(n-s) \times s$, and $(n-s) \times (n-s)$, respectively. But, by (4), for i, j, $1 \leqslant i, j \leqslant s$, we have

$$\begin{aligned} A_{ij} &= \langle e^*_{g_i\cdot\alpha} \mid e^*_{g_j\cdot\alpha} \rangle \\ &= \begin{cases} 0 & \text{if } i \neq j, \\ \frac{\chi(1)}{|G|} \sum\limits_{\sigma \in G_\alpha} \chi(\sigma) & \text{if } i = j, \end{cases} \\ &= \begin{cases} 0 & \text{if } i \neq j, \\ \frac{1}{[G:G_\alpha]} \dim V_\alpha^* & \text{if } i = j, \end{cases} \\ &= \begin{cases} 0 & \text{if } i \neq j, \\ s/n & \text{if } i = j, \end{cases} \\ &= \left(\frac{s}{n} I_s\right)_{ij}, \end{aligned}$$

where I_s is the $s \times s$ identity matrix. So the block form of A has been changed to the form

$$A = \begin{bmatrix} \frac{s}{n} I_s & A_2 \\ A_3 & A_4 \end{bmatrix}.$$

We now claim that A is idempotent. In fact,

$$\begin{aligned}
(A^2)_{ij} &= \sum_{\ell=1}^{n} A_{i\ell} A_{\ell j} \\
&= \sum_{\ell=1}^{n} \langle e^*_{g_i \cdot \alpha} \mid e^*_{g_\ell \cdot \alpha} \rangle \langle e^*_{g_\ell \cdot \alpha} \mid e^*_{g_j \cdot \alpha} \rangle \\
&= \sum_{\ell=1}^{n} \left(\frac{\chi(1)}{|G|} \sum_{\sigma \in G_\alpha} \chi(g_\ell \sigma g_i^{-1}) \right) \left(\frac{\chi(1)}{|G|} \sum_{\tau \in G_\alpha} \chi(g_j \tau g_\ell^{-1}) \right) \\
&= \frac{\chi(1)^2}{|G|^2} \sum_{\ell=1}^{n} \sum_{\sigma \in G_\alpha} \sum_{\tau \in G_\alpha} \chi(g_\ell \sigma g_i^{-1}) \chi(g_j \tau g_\ell^{-1}) \\
&= \frac{\chi(1)^2}{|G|^2} \sum_{\ell=1}^{n} \sum_{\lambda \in g_\ell G_\alpha} \sum_{\mu \in G_\alpha g_\ell^{-1}} \chi(\lambda g_i^{-1}) \chi(g_j \mu) \\
&= \frac{\chi(1)^2}{|G|^2} \sum_{\ell=1}^{n} \sum_{\lambda \in g_\ell G_\alpha} \sum_{\sigma \in G_\alpha} \chi(\lambda g_i^{-1}) \chi(g_j \sigma \lambda^{-1}) \\
&= \frac{\chi(1)^2}{|G|^2} \sum_{\lambda \in G} \sum_{\sigma \in G_\alpha} \chi(\lambda g_i^{-1}) \chi(g_j \sigma \lambda^{-1}) \\
&= \frac{\chi(1)^2}{|G|^2} \sum_{\lambda \in G} \sum_{\sigma \in G_\alpha} \chi(\lambda) \chi(g_j \sigma g_i^{-1} \lambda^{-1}) \\
&= \frac{\chi(1)^2}{|G|^2} \sum_{\sigma \in G_\alpha} \sum_{\lambda \in G} \chi(\lambda) \chi(g_j \sigma g_i^{-1} \lambda^{-1}) \\
&= \frac{\chi(1)^2}{|G|^2} \sum_{\sigma \in G_\alpha} \left(\frac{|G|}{\chi(1)} \chi(g_j \sigma g_i^{-1}) \right) \\
&= \frac{\chi(1)}{|G|} \sum_{\sigma \in G_\alpha} \chi\left(g_j \sigma g_i^{-1} \right) \\
&= \langle e^*_{g_i \cdot \alpha} \mid e^*_{g_j \cdot \alpha} \rangle \\
&= A_{ij}.
\end{aligned}$$

Therefore, the claim holds. Now, using $A^2 = A$, we obtain

$$A_2 A_3 = \left(\frac{s}{n} - \frac{s^2}{n^2} \right) I_s.$$

Since $0 < s < n$, $A_2 A_3$ is an invertible matrix and hence $s \leqslant n - s$, $s \leqslant \frac{1}{2}n$, or $\chi(1)(\chi, \theta)_G \leqslant \frac{1}{2}n$, which is a contradiction. Thus $V_\chi^n(G)$ does not have an O-basis. $\square$

Corollary 1 (Main theorem [3]). *Let G be a* 2*-transitive subgroup of* $\mathbb{S}_n$, $n \geqslant 3$. *Let* $\chi = \theta - 1_G$, *where* θ *is the permutation character of G. If* $\dim V = m \geqslant 2$, *then* $V_\chi^n(G)$ *does not have an O-basis.*

Proof. G has a natural transitive and faithful action on the set $\Omega = \{1, 2, \ldots, n\}$, given by $\sigma \cdot i = \sigma(i)$. 2-Transitivity of this action implies that χ is an irreducible constituent of θ. On the other hand, $n \geqslant 3$ and so $\chi(1)(\chi, \theta)_G = (n-1) > \frac{1}{2}n$. Hence, by our main theorem, $V_\chi^n(G)$ does not have an O-basis. $\square$

Example. Let $G = \Omega = \mathbb{A}_4$ be the alternating group of degree 4. We know that G acts transitively and faithfully on Ω by right multiplication, so G is a subgroup of $\mathbb{S}_{12}$. G has an irreducible complex character, say χ, of degree 3. As θ, the permutation character of G, is regular, χ is an irreducible constituent of θ of multiplicity 3. Therefore, $\chi(1)(\chi, \theta)_G = 9 > 6 = \frac{1}{2}|\Omega|$, so by the main theorem $V_\chi^{12}(G)$ does not have an O-basis. Note that in this example the action of G on Ω is not 2-transitive, so our main result really extends the result of [3].

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